

THE ZAPPA-SZEP PRODUCT OF LEFT-ORDERABLE GROUPS

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ABSTRACT. It is well-known that the direct product of left-orderable groups is left-orderable and that, under a certain condition, the semi-direct product of left-orderable groups is left-orderable. We extend this result and show that, under a similar condition, the Zappa-Szep product of left-orderable groups is left-orderable. Moreover, we find conditions that ensure the existence of a partial left and right invariant ordering (bi-order) in the Zappa-Szep product of bi-orderable groups and prove some properties satisfied.

INTRODUCTION

Let G be a group with subgroups H and K such that $G = HK$ and $H \cap K = \{1\}$. Then G is isomorphic to the Zappa-Szep product of H and K , denoted by $H \rtimes K$. If both subgroups H and K are normal, G is their direct product $H \times K$ and if H only is normal, G is their semi-direct product $H \rtimes K$. The Zappa-Szep product of groups is a generalisation of the direct and the semi-direct product which requires the embedding of neither of the factors to be normal in the product. We recall that a group G is *left-orderable* if there exists a total ordering $<$ of its elements which is invariant under left multiplication, that is $g < h$ implies $fg < fh$ for all f, g, h in G . It is well-known that the direct product of left-orderable groups is left-orderable and that, under a certain condition, the semi-direct product of left-orderable groups is left-orderable [9], [8][p.27]. We extend this result and show that, under a similar condition, the Zappa-Szep product of left-orderable groups is left-orderable. Before we state our first result, we introduce some notations.

Definition. Let H, K be groups. Let α be the homomorphism defined by $\alpha : K \rightarrow \text{Sym}(H)$, $k \mapsto \alpha_k$, where $k \in K$ and $\text{Sym}(H)$ denotes the group of bijections of H . Let $\beta : H \rightarrow \text{Sym}(K)$, $h \mapsto \beta_h$, be such that $\beta_{h_1 h_2} = \beta_{h_2} \circ \beta_{h_1}$ and $\beta_1 = \text{Id}_K$ (β an anti-homomorphism). Assume (α, β) satisfies: $\alpha_k(h_1 h_2) = \alpha_k(h_1) \alpha_{\beta_{h_1}(k)}(h_2)$ and $\beta_h(k_1 k_2) = \beta_{\alpha_{k_2}(h)}(k_1) \beta_h(k_2)$. The Zappa-Szep product $H \rtimes K$ of H and K with respect to the pair (α, β) is the set $H \times K$ endowed with the following product:

$$(h_1, k_1) (h_2, k_2) = (h_1 \alpha_{k_1}(h_2), \beta_{h_2}(k_1) k_2)$$

The identity is $(1, 1)$ and the inverse of an element (h, k) is

$$(h, k)^{-1} = (\alpha_{k^{-1}}(h^{-1}), \beta_{h^{-1}}(k^{-1}))$$

In fact, $H \times \{1\}$ and $K \times \{1\}$ are subgroups of $H \rtimes K$, isomorphic to H and K , respectively.

The Zappa-Szep product is also called crossed product, bi-crossed product, knit product, two-sided semidirect product. We refer to [3], [1], [13], [14] for details. Note that every element in $H \rtimes K$ is uniquely represented by hk , with $h \in H$ and $k \in K$, and kh is equal to $\alpha_k(h) \beta_h(k)$ in $H \rtimes K$.

Theorem 1. Let G be the Zappa-Szep product of the groups H and K with respect to (α, β) . Assume H and K are left-orderable groups. Let P_H and P_K denote the positive cones (of a left order) of H and K respectively. Assume the following condition $(*)$ is satisfied:

$$\alpha_k(P_H) \subseteq P_H, \forall k \in K$$

Then, there exists a total left order \prec on $G = H \rtimes K$, with positive cone P , such that (the embedding of) K is a convex subgroup with respect to \prec and $P_K = P \cap K$.

We introduce some definitions and refer to [2], [6], [7], [8], [9], [10], [11], [12]. Let G be a left-orderable group with a strict total left order \prec . An element g , $g \in G$, is called *positive* if $1 \prec g$ and the set of positive elements P is called the *positive cone* of \prec . The positive cone P satisfies:

- (1) P is a semigroup, that is $P \cdot P \subseteq P$
- (2) G is partitioned, that is $G = P \cup P^{-1} \cup \{1\}$ and $P \cap P^{-1} = \emptyset$

Conversely, if there exists a subset P of G that satisfies (1) and (2), then P determines a unique total left order \prec defined by $g \prec h$ if and only if $g^{-1}h \in P$. If P satisfies only (1), the left order obtained is partial. A subgroup N of a left-orderable group G is called *convex* if for any $x, y, z \in G$ such that $x, z \in N$ and $x \prec y \prec z$, we have $y \in N$.

Proof. of Theorem 1 Let $P \subseteq G$ be defined by: $g = hk \in P$ if $h \in P_H$ or if $h = 1$, $k \in P_K$. We show there exists a total left order \prec on $G = H \rtimes K$ with positive cone P . First, we prove P is a semigroup. Let $g = hk \in P$ and $g' = h'k' \in P$. If $h' = 1$ and $h = 1$, then $k, k' \in P_K$ and $kk' \in P_K$, since P_K is a semigroup, so $gg' \in P$. If $h' = 1$ and $h \neq 1$, then $h \in P_H$ and so $gg' = hkk' \in P$. If $h' \neq 1$, then $h' \in P_H$ and $gg' = hkh'k' = h\alpha_k(h')\beta_{h'}(k)k'$. From the assumption (*), $\alpha_k(h') \in P_H$, so if $h = 1$, then $gg' = \alpha_k(h')\beta_{h'}(k)k' \in P$ and if $h \neq 1$, $h\alpha_k(h') \in P_H$ also, since P_H is a semigroup, so $gg' \in P$. Next, we prove that given $g = hk \neq 1$ in $G = H \rtimes K$, either g belongs to P or $g^{-1} = \alpha_{k^{-1}}(h^{-1})\beta_{h^{-1}}(k^{-1})$ belongs to P . Assume $h = 1$. If $k \in P_K$, then $g \in P$, otherwise $k^{-1} \in P_K$, since P_K partitions K , and then $g^{-1} = k^{-1} \in P$. Furthermore, g cannot belong to $P \cap P^{-1}$, since it would contradict $P_K \cap P_K^{-1} = \emptyset$. Assume $h \neq 1$. If $h \in P_H$, then $g \in P$, otherwise $h^{-1} \in P_H$, since P_H partitions H , and from (*), $\alpha_{k^{-1}}(h^{-1}) \in P_H$. So, $g^{-1} = \alpha_{k^{-1}}(h^{-1})\beta_{h^{-1}}(k^{-1})$ belongs to P . Assume $g \in P \cap P^{-1}$, then $h \in P_H$ and $\alpha_{k^{-1}}(h^{-1}) \in P_H$. It holds that $\alpha_{k^{-1}}(h^{-1}) = (\alpha_{\beta_{h^{-1}}(k^{-1})}(h))^{-1}$. Indeed, on one side $\alpha_{k^{-1}}(1) = 1$ and on the other side $\alpha_{k^{-1}}(1) = \alpha_{k^{-1}}(h^{-1}h) = \alpha_{k^{-1}}(h^{-1})\alpha_{\beta_{h^{-1}}(k^{-1})}(h)$. From (*), $\alpha_{\beta_{h^{-1}}(k^{-1})}(h)$ belongs to P_H , so $\alpha_{k^{-1}}(h^{-1}) = (\alpha_{\beta_{h^{-1}}(k^{-1})}(h))^{-1}$ belongs to $P_H \cap P_H^{-1}$ and this is a contradiction. So, P satisfies the conditions (1) and (2) and it determines uniquely a total left order \prec . Let $g = hk \in G$ and $k' \in K$. Assume $1 \prec g \prec k'$ and assume by contradiction that $h \neq 1$. From $1 \prec g$, we have $h \in P_H$. From $hk \prec k'$, we have $k'^{-1}hk \prec 1$. But, $k'^{-1}hk = \alpha_{k'^{-1}}(h)\beta_h(k'^{-1})k \succ 1$, since, from (*), $\alpha_{k'^{-1}}(h)$ belongs to P_H . So, $h = 1$. \square

In the following example, we remark it is sometimes useful to consider a semi-direct product as a special case of Zappa-Szep product in order to show it is left-orderable. Indeed, this permits to check condition (*) either on a positive cone of H or on a positive cone of K . Let $H = \text{Gp}\langle y, z \mid y^3 = z^3 \rangle$. Since $H = \mathbb{Z} *_3 \mathbb{Z}$ and \mathbb{Z} is (left) orderable, H is left-orderable [8][p.178]. Let $K = \text{Gp}\langle a, b \mid aba = bab \rangle$, the braid group on 3 strands, K is left-orderable [5]. Let $\alpha : K \rightarrow \text{Sym}(H)$ be the trivial homomorphism and let $\beta : H \rightarrow \text{Sym}(K)$, the anti-homomorphism defined by $\beta_y = \beta_z = (a, b)$. With respect to (α, β) , let $G = H \rtimes K$ in this specific order; G is presented by $\text{Gp}\langle y, z, a, b \mid aba = bab, y^3 = z^3, ay = yb, by = ya, az = zb, bz = za \rangle$. We show G is left-orderable, using Thm.1. Each element $h \in H$ admits a normal form $h = y^{\epsilon_1} z^{\mu_1} y^{\epsilon_2} z^{\mu_2} \dots y^{\epsilon_m} z^{\mu_m} \Delta^{n_h}$, where $\Delta = y^3 = z^3$ is central in H , $-1 \leq \epsilon_i, \mu_i \leq 1$ and $n_h \in \mathbb{Z}$. We define $\exp(h) = \epsilon_1 + \mu_1 + \dots + \epsilon_m + \mu_m + 3n_h$ and P_H to be the set of elements $h \in H \setminus \{1\}$ such that $\exp(h) > 0$, or if $\exp(h) = 0$, $n_h > 0$, or if $\exp(h) = n_h = 0$, $\epsilon_1 \neq 0$. The set P_H is a positive cone (the proof appears in the appendix) and P_H satisfies trivially the condition (*) from Thm.1. So, G is left-orderable.

Note that a group is left-orderable if and only if it is right-orderable. This is well illustrated here with the existence of a symmetric version of this result. Indeed, if H and K are right-orderable groups, with Q_H and Q_K positive cones of right orders of H and K

satisfying $\beta_h(Q_K) \subseteq Q_K$, $\forall h \in H$. Then there exists a total right order $<$ on $G = H \rtimes K$, with positive cone Q (defined by $g = hk \in Q$ if $k \in Q_K$ or if $k = 1$, $h \in Q_H$), such that (the embedding of) H is a convex subgroup with respect to $<$. We say a group G is *partially (totally) bi-orderable* if there exists a partial (total) ordering \ll of its elements which is invariant under left and right multiplication, that is $g \ll h$ implies $fgk \ll fhk$ for all f, g, h, k in G . In particular, a set P determines a partial bi-order if and only if P is a semigroup, and satisfies $gPg^{-1} \subseteq P$ for all $g \in G$; P determines a total bi-order if additionally $G = P \cup P^{-1} \cup \{1\}$ and $P \cap P^{-1} = \emptyset$. A natural question is when the Zappa-Szep product of bi-orderable groups is a bi-orderable group. In the following theorem, we give conditions that ensure the existence of a partial bi-order.

Theorem 2. *Let G be the Zappa-Szep product of the groups H and K with respect to (α, β) . Assume H and K are partially bi-orderable groups. Let P_H and P_K denote the positive cones (of a partial bi-order) of H and K respectively. Assume the following conditions are satisfied:*

(*) $\alpha_k(P_H) \subseteq P_H$, $\forall k \in P_K$, and $\beta_h(P_K) \subseteq P_K$, $\forall h \in P_H$

(**) $kP_Hk^{-1} \subseteq P_H$, $\forall k \in K$, and $hP_Kh^{-1} \subseteq P_K$, $\forall h \in H$

Then, there exists a partial bi-order \ll on $G = H \rtimes K$. Furthermore, if $h, h' \in H$ satisfy $h \ll h'$ then $\alpha_k(h) \ll \alpha_k(h')$ and $\beta_h(k) \ll \beta_h(k')$, $\forall k \in K$ and if $k, k' \in K$ satisfy $k \ll k'$ then $\alpha_k(h) \ll \alpha_{k'}(h)$ and $\beta_h(k) \ll \beta_h(k')$, $\forall h \in H$.

Proof. Let $P \subseteq G$ be defined by: $g = hk \in P$ if $h \in P_H$ and $k \in P_K$; if $h = 1$, $k \in P_K$ or if $k = 1$, $h \in P_H$. We show P is a semigroup and $gPg^{-1} \subseteq P$ for all $g \in G$. Let $g = hk \in P$ and $g' = h'k' \in P$, then $gg' = hkh'k' = h\alpha_k(h')\beta_{h'}(k)k'$. From the assumption (*), $\alpha_k(h') \in P_H$ and $\beta_{h'}(k) \in P_K$, so $h\alpha_k(h') \in P_H$ and $\beta_{h'}(k)k' \in P_K$, since P_H and P_K are semigroups, so $gg' \in P$. Next, let $g = hk \in G$ and $h'k' \in P$, we show $gh'k'g^{-1} \in P$: $gh'k'g^{-1} = hkh'h'k'^{-1}h^{-1} = (h(kh'h'k'^{-1})h^{-1})(h(kk'k^{-1})h^{-1})$. From (**), $h' \in P_H$ implies $kh'h'k'^{-1} \in P_H$ and so $h(kh'h'k'^{-1})h^{-1} \in P_H$, since P_H is the positive cone of a bi-order. The element $h(kk'k^{-1})h^{-1} \in P_K$, first from $kk'k^{-1} \in P_K$, and next from (**).

Let $h, h' \in H$ satisfy $h \ll h'$, and let $k \in K$, then $kh \ll kh'$, that is $\alpha_k(h)\beta_h(k) \ll \alpha_k(h')\beta_{h'}(k)$. Since \ll is a bi-order, this implies $1 \ll (\alpha_k(h))^{-1}\alpha_k(h')\beta_{h'}(k)(\beta_h(k))^{-1}$ and from the definition of \ll , we have $(\alpha_k(h))^{-1}\alpha_k(h') \in P_H$ and $\beta_{h'}(k)(\beta_h(k))^{-1} \in P_K$, that is $\alpha_k(h) \ll \alpha_k(h')$ and $\beta_h(k) \ll \beta_{h'}(k)$. In case $h = 1$ $h' \neq 1$, this means $\alpha_k(P_H) \subseteq P_H$, $\forall k \in K$ and $h' \gg 1$ implies $\beta_{h'}(k) \gg k$, $\forall k \in K$; in case $h \neq 1$ $h' = 1$, this means that $h \ll 1$ implies $\alpha_k(h) \ll 1$ and $\beta_h(k) \ll k$, $\forall k \in K$. The same proof works for the symmetric statement. \square

It would be interesting to know if there exists a natural extension of total bi-orders of H and K that defines a total bi-order on $H \rtimes K$ and $H \rtimes K$ is not a direct nor a semi-direct product.

APPENDIX

We recall each element $h \in H$ admits a normal form $h = y^{\epsilon_1}z^{\mu_1}y^{\epsilon_2}z^{\mu_2}...y^{\epsilon_m}z^{\mu_m}\Delta^{n_h}$, where $\Delta = y^3 = z^3$ is a central element in H , $-1 \leq \epsilon_i, \mu_i \leq 1$ and $n_h \in \mathbb{Z}$. We define $\exp(h) = \epsilon_1 + \mu_1 + .. + \epsilon_m + \mu_m + 3n_h$ and P_H to be the set of elements $h \in H \setminus \{1\}$ such that $\exp(h) > 0$ (class (A)), or if $\exp(h) = 0$, $n_h > 0$ (class (B)), or if $\exp(h) = n_h = 0$, $\epsilon_1 \neq 0$ (class (C)). We show P_H is a positive cone. First, we show P_H is a semigroup. Let $h = y^{\epsilon_1}z^{\mu_1}...y^{\epsilon_m}z^{\mu_m}\Delta^{n_h}$, $h' = y^{\epsilon'_1}z^{\mu'_1}...y^{\epsilon'_m}z^{\mu'_m}\Delta^{n_{h'}}$, with $-1 \leq \epsilon_i, \mu_i, \epsilon'_i, \mu'_i \leq 1$, be elements of P_H . We need to check several cases in order to show $hh' \in P_H$.

Case 1: if h is in class (A) and h' is in class (A) or (B) or (C), then hh' is also in (A).

Case 2: if h, h' are in class (B), then $hh' \in P_H$, since $\exp(hh') = 0$ and $n_{hh'} \geq n_h + n_{h'} - 1 > 0$. Indeed, there are only two cases in which $n_{hh'}$ decreases: if h ends with y^{-1} and h' begins with y^{-1} , or if h ends with z^{-1} and h' begins with z^{-1} , and in both cases $n_{hh'} = n_h + n_{h'} - 1$.

Case 3: assume h, h' are in class (C). Note that if $h = y^{\epsilon_1} z^{\mu_1} \dots y^{\epsilon_m} z^{\mu_m} \Delta^{n_h}$, $-1 \leq \epsilon_i, \mu_i \leq 1$, satisfies $\exp(h) = n_h = 0$, then the length of h is necessarily even. Furthermore, if h is in class (C), then $h = y^{\pm 1} z^{\mu_1} y^{\epsilon_2} z^{\mu_2} \dots y^{\epsilon_m} z^{\pm 1}$ and the same holds for h' . So, hh' is also in (C). Case 4: if h is in class (B) and h' is in class (C), then $h = y^{\epsilon_1} z^{\mu_1} \dots y^{\epsilon_m} z^{\mu_m} \Delta^{n_h}$, $n_h > 0$ and $h' = y^{\pm 1} z^{\mu'_1} \dots y^{\epsilon'_m} z^{\pm 1}$. It holds that $n_h - 1 \leq n_{hh'} \leq n_h + 1$ and we need only to check the case $n_{hh'} = n_h - 1$. If $n_h - 1 > 0$, then hh' is in class (B). If $n_h - 1 = 0$, then $h = y^{\epsilon_1} z^{\mu_1} \dots z^{\mu_{m-1}} y^{-1} \Delta$, $h' = y^{-1} z^{\mu'_1} \dots y^{\epsilon'_m} z^{\pm 1}$ and $hh' = y^{\epsilon_1} z^{\mu_1} \dots z^{\mu_{m-1}} y z^{\mu'_1} \dots y^{\epsilon'_m} z^{\pm 1}$. We show hh' is in class (C), that is $\epsilon_1 \neq 0$. On one-hand, $\exp(h) = \epsilon_1 + \mu_1 + \dots + \mu_{m-1} - 1 + 3$ and on the second-hand $\exp(h) = 0$. By induction on m , it holds that $\epsilon_1 + \mu_1 + \dots + \epsilon_{m-1} + \mu_{m-1} = -2$, with $-1 \leq \epsilon_i, \mu_i \leq 1$, implies $\epsilon_1 \neq 0$.

Case 5: if h is in class (C) and h' is in class (B), then $h = y^{\pm 1} z^{\mu_1} \dots y^{\epsilon_m} z^{\pm 1}$, and $h' = y^{\epsilon'_1} z^{\mu'_1} \dots y^{\epsilon'_m} z^{\mu'_m} \Delta^{n_{h'}}$, $n_{h'} > 0$. It holds that $n_{h'} - 1 \leq n_{hh'} \leq n_{h'} + 1$ and in case $n_{hh'} = n_{h'} - 1 = 0$, hh' is necessarily in class (C), as hh' begins with $y^{\pm 1}$.

Next, we show P_H partitions H , that is given $h \in H \setminus \{1\}$, either $h \in P_H$ or $h^{-1} \in P_H$. If $\exp(h) > 0$, then $h \in P_H$, otherwise assume $\exp(h) \leq 0$. If $\exp(h) < 0$, then $h^{-1} \in P_H$, since $\exp(h^{-1}) = -\exp(h)$. So, assume $\exp(h) = 0$. If $n_h > 0$, then $h \in P_H$, otherwise assume $n_h \leq 0$. If $n_h < 0$, then $h^{-1} \in P_H$, since $n_{h^{-1}} = -n_h$. So, assume $n_h = 0$. If $h = y^{\epsilon_1} z^{\mu_1} \dots y^{\epsilon_m} z^{\mu_m} \Delta^{n_h}$, $-1 \leq \epsilon_i, \mu_i \leq 1$, satisfies $\exp(h) = n_h = 0$, then there are two possibilities: either $h = y^{\pm 1} z^{\mu_1} y^{\epsilon_2} z^{\mu_2} \dots y^{\epsilon_m} z^{\pm 1}$ or $h = z^{\pm 1} y^{\epsilon_2} z^{\mu_2} \dots y^{\epsilon_{m-1}} z^{\mu_{m-1}} y^{\pm 1}$. If $h = y^{\pm 1} z^{\mu_1} y^{\epsilon_2} z^{\mu_2} \dots y^{\epsilon_m} z^{\pm 1}$, then $h \in P_H$. Otherwise, if $h = z^{\pm 1} y^{\epsilon_2} z^{\mu_2} \dots y^{\epsilon_{m-1}} z^{\mu_{m-1}} y^{\pm 1}$, then $h^{-1} = y^{\mp 1} z^{-\mu_{m-1}} y^{-\epsilon_{m-1}} z^{-\mu_2} y^{-\epsilon_2} \dots z^{\mp 1}$, that is $h^{-1} \in P_H$.

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